Solution of Highly Constrained Optimal Control Problems Using Nonlinear Programing

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Theme

THIS paper considers the solution of highly constrained optimal control problems using the nonlinear programing method of Fiacco-McCormick.[1] Several authors[2,3] have successfully applied the technique to constrained optimal control problems of a limited scope. The present paper expands the theory to encompass a general mathematical model for trajectory optimization capable of directly handling six types of equality and inequality constraints. The user-oriented model is designed to facilitate the rapid set up of a wide range of different simulations and provides for the simultaneous optimization of design parameters and continuous control variables. Accurate and efficient methods of unconstrained function minimization and linear search required to implement the Fiacco-McCormick method are discussed.

Contents

The general mathematical model presented provides a flexible skeletal framework for describing a wide spectrum of complex optimal control problems in terms of problem-oriented functions. The model is capable of incorporating two classes of independent variables which are to be chosen to extremize some objective function. Independent variables which are functions of time are termed dynamic control variables and are designated by $u_k(t)$. Independent variables which are constant with respect to time are termed design variables, d_p. Trajectory sectioning is a device commonly used to provide flexibility in modeling. It is a method of subdividing the time history of a trajectory simulation into parts relevant to the description of the simulation. A section is defined as any portion of the trajectory in which the mathematical model is of a given form and the state variables $x_i(t)$ are continuous functions of time. Section endpoints are chosen to coincide with points at which the differential equations of motion, the control model, or the trajectory constraints change form; or at which the state variables experience a discontinuity. If the subscript j denotes the trajectory section, then the general optimal control problem is to

$$\underset{u_k(t),d_p}{\text{minimize}}\ J = \Phi(t_j^{\,0}, t_j^{\,f}, x_{ij}^{\,0}, x_{ij}^{\,f}, d_p)+\sum_{j=1}^{l}\int_{t_j^0}^{t_j^f} L_j[x_{ij}(t), u_{kj}(t), d_p, t]\,dt \tag{1}$$

subject to the differential equations

$$\dot{x}_{ij}(t) = f_{ij}[x_{ij}(t), u_{kj}(t), d_p, t] \qquad \begin{array}{l} i = 1, 2, \ldots, n \\ k = 1, 2, \ldots, w \\ j = 1, 2, \ldots, l \end{array} \tag{2}$$

and constraints

$$\xi_{sj}[x_{ij}(t), u_{kj}(t), d_p, t] \geqq 0 \qquad s = 1, 2, \ldots, n_\xi \tag{3}$$

$$\eta_{sj}[x_{ij}(t), u_{kj}(t), d_p, t] = 0 \qquad s = 1, 2, \ldots, n_\eta \tag{4}$$

$$\zeta_s(t_j^{\,0}, t_j^{\,f}, x_{ij}^{\,0}, x_{ij}^{\,f}, d_p) \geqq 0 \qquad s = 1, 2, \ldots, n_\zeta \tag{5}$$

$$\Psi_s(t_j^{\,0}, t_j^{\,f}, x_{ij}^{\,0}, x_{ij}^{\,f}, d_p) = 0 \qquad s = 1, 2, \ldots, n_\Psi \tag{6}$$

$$\sum_{j=1}^{l}\int_{t_j^0}^{t_j^f} Q_{sj}[x_{ij}(t), u_{kj}(t), d_p, t]\,dt - D_s \geqq 0 \qquad s = 1, 2, \ldots, n_Q \tag{7}$$

$$\sum_{j=1}^{l}\int_{t_j^0}^{t_j^f} P_{sj}[x_{ij}(t), u_{kj}(t), d_p, t]\,dt - C_s = 0 \qquad s = 1, 2, \ldots, n_P \tag{8}$$

where $\xi_{sj}, \eta_{sj}, Q_{sj}$ and P_{sj} denote independent functions applicable during trajectory section j. Superscripts "0" and "f" denote evaluation of initial and final times of corresponding sections. Eqs. (1) and (2) (subject to Constraints 3 to 8) comprise the general optimization model. Dynamic inequality constraints, $\xi_{sj} \geqq 0$, include the familiar state and control variable inequality constraints as a subset. The parametric equality constraints Ψ_s allow specification of the boundary conditions on the differential equations, as well as constraints on the design variables.

The design parameters, d_p may serve a variety of useful purposes apart from modeling obvious design parameters (wing area, sweepback angle). Undetermined initial conditions on state variables at the start of any section may be treated as design variables.[4] Thus, problems whose initial state is confined to some subset of state space by equality and inequality constraints [Eqs. (5) and (6)] are easily treated. The time duration of any section may be treated as a design parameter. A more detailed discussion of this model may be found in the full paper.

Nonlinear programing is a term applied to any algorithmic procedure seeking to extremize a function of N independent variables which are restricted to some subspace of Euclidian N-space. The nonlinear programing problem seeks the minimum of $J(y)$ subject to the constraints $g_i(y) \geqq 0$ and $h_j(y) = 0$. To solve the nonlinear programing problem, Fiacco and McCormick have developed the Sequential Unconstrained Minimization Technique (SUMT).[1] This algorithm seeks the solution to the above constrained minimization problem as the limit of a sequence of unconstrained minimization problems. At each stage a value for the parameter r_k is selected and the penalty function

$$\mathbf{P}(y, r_k) = J(y) + r_k \sum_{i=1}^{m} \frac{1}{g_i(y)} + r_k^{-1/2} \sum_{j=1}^{p} h_j^{\,2}(y) \tag{9}$$

is minimized using any one of a number of function minimization algorithms.[5-7] In the limit, as $r_k \to 0$, minimizing the penalty function minimizes the performance index, drives the equality constraints into satisfaction, and allows the inequality constraints to approach as close to zero as may be optimal. A sequence of minimizations is performed with $r_1 > r_2 > \cdots > r_k > \cdots > r_f$ rather than just one minimization of $\mathbf{P}(y, r_f)$ because the latter minimization problem is very difficult to solve from a numerical

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standpoint. Starting with a large r_k results in a relatively easy minimization problem. Minimization of the function **P** for $r = r_k$ then provides a good initial estimate for the minimization of the subsequent problem with $r = r_{k+1}$.

The trajectory optimization problem is a nonlinear programing problem except for one feature: a continuous function of time is sought as the solution to the trajectory optimization problem, whereas the solution to the nonlinear programing problem is represented by a point in Euclidian N-space. This dissimilarity is resolved by approximating the continuous function of time by a function of n independent parameters. The dynamic optimal control can be approximated to any desired accuracy by refining the parameterization of the control function.

The penalty function for the general trajectory optimization problem is formed in a fashion analogous to Eq. (9). Before exhibiting the function **P** it is convenient to transform the integral constraints, Eqs. (7) and (8), into parametric constraints on the final values of the following pseudo-state variables:

$$\dot{x}_{(n+i),j} = P_{ij} \qquad i = 1, 2, \ldots, n_p \tag{10}$$

$$\dot{x}_{(n+n_p+i),j} = Q_{ij} \qquad i = 1, 2, \ldots, n_Q \tag{11}$$

with corresponding parametric constraints

$$x^f_{(n+i),l} - C_i = 0 \tag{12}$$

$$x^f_{(n+n_p+i),l} - D_i \geqq 0 \tag{13}$$

With these definitions the penalty function may be written

$$\mathbf{P} = G(t_j{}^0, t_j{}^f, x_{ij}{}^0, x_{ij}{}^f, d_p) + \sum_{j=1}^{l} \int_{t_j{}^0}^{t_j{}^f} L_j{}^*[x_{ij}(t), u_{kj}(t), d_p, t]\, dt \tag{14}$$

where

$$G = \Phi + r^{-1/2} \sum_{s=1}^{n_\Psi} \Psi_s{}^2 + r^{-1/2} \sum_{i=1}^{n_p} [x^f_{(n+i),l} - C_i]^2 + r \sum_{s=1}^{n_\zeta} 1/\zeta_s + r \sum_{i=1}^{n_Q} 1/[x^f_{(n+n_p+i),l} - D_i] \tag{15}$$

and

$$L_j{}^* = L_j + r^{-1/2} \sum_{s=1}^{n_\eta} \eta_{sj}{}^2 + r \sum_{s=1}^{n_\xi} 1/\xi_{sj} \tag{16}$$

The study of the technique presented here has resulted, in part, in a general purpose computer program that includes all of the features discussed. The program uses either the method of Fletcher-Reeves[5] or the method of Fletcher-Powell[6] to perform the required unconstrained function minimization. The one-dimensional search required for function minimization is performed using a variation of the golden section search with cubic fit.[7] The gradients required may be computed by forward finite differences, symmetric finite differences, or a parametric variation of the impulsive response method.[4] Details of the numerical techniques are discussed in the full paper. Several examples, including a hypersonic re-entry problem, are also presented.

The method of nonlinear programing is a powerful tool for the solution of highly constrained trajectory optimization problems. Since the formulation presented incorporates all constraints and boundary conditions in penalty terms, no new variational analysis need be performed to implement new constraints or even new problems: only the functions Φ, L, f_i, ξ, η, ζ, Ψ, Q and P [Eqs. (1–8)] and their first partial derivatives need be provided to the optimization program. Indirect methods usually require a substantial amount of variational analysis to implement new problems, or even new constraints or boundary conditions.

The penalty paid for this advantage is an increased computational load over what might be achieved by an indirect method tailored to a particular problem. This increased computational cost must be weighed against the cost of the preliminary analysis and reprograming needed to implement an indirect technique. This analysis is particularly complex for the variational treatment of state variable inequality constraints for multiple-constrained optimization problems.

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